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Exactly solvable models with $\mathcal{PT}$-symmetry and with an asymmetric coupling of channels

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Abstract
Bound states generated by the K coupled $\mathcal{PT}$-symmetric square wells are studied in a series of models where the Hamiltonians are assumed $\mathcal{R}$-pseudo-Hermitian and $\mathcal{R}^2$-symmetric. Specific rotation-like generalized parities $\mathcal{R}$ are considered such that $\mathcal{R}^N = I$ at some integers N. We show how our assumptions make the models exactly solvable and quasi-Hermitian. This means that they possess the real spectra as well as the standard probabilistic interpretation.

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1. Introduction

Bender's and Boettcher's $\mathcal{PT}$-symmetric version of quantum mechanics [1] admits a transition to complex potentials $V(x)$ (say, on a finite interval) characterized by the $\mathcal{PT}$-symmetry property $\mathcal{PT}V(x) = V(x)\mathcal{PT}$ where $\mathcal{P}$ denotes parity while the complex conjugation $\mathcal{T}$ mimics time reversal. One of the simplest illustrative examples of the corresponding non-Hermitian $\mathcal{PT}$-symmetric oscillator with real spectrum is generated by the purely imaginary square-well potential step [2]

$$V(x) = V_{(Z)}(x) = -\mathrm{i}Z\,\mathrm{sign}(x), \qquad x \in (-1, 1). \tag{1}$$

The solvability of this model facilitates an introduction of the norm of its wavefunctions in a suitable metric, i.e., via an introduction of a Hamiltonian-dependent scalar product [3]. It also renders possible the correct transition to classical limit [4]. Interesting physical applications of equation (1) were found in supersymmetric context [5] as well as beyond quantum theory where schematic equation (1) and its modifications may play role in an explanation of the mode-swapping phenomena in classical magnetohydrodynamics [6]. In the mathematical context, model (1) helped to clarify the mechanisms of the spontaneous breakdown of $\mathcal{PT}$-symmetry at a critical strength $Z = Z_{\mathrm{crit}}$ of non-Hermiticity [7, 8].

A specific merit of model (1) lies in the feasibility of a transition to its more sophisticated piecewise constant solvable alternatives [9]. A particularly promising new direction of development has recently been found in the tentative use of the elementary functions (1) as forces which mediate an interaction between two [10] and/or three [11] coupled square-well oscillators. Here we intend to move one step further and to reanalyse the similar coupled K-channel problems in a more systematic manner.

Our key idea lies in the observation that in the one-dimensional Schrödinger equation which describes K coupled channels,

$$-\frac{d^2}{dx^2}\varphi^{(m)}(x)+\sum_{j=1}^{K} V_{Z_{(m,j)}}(x)\varphi^{(j)}(x)=E\varphi^{(m)}(x), \qquad m=1,2,\ldots,K, \tag{2}$$

the complications connected with its solution grow very quickly with K. In general, the properties of the model are controlled by as many as K^2 independent real couplings $Z_{(m,j)}$ in equation (1). An introduction of some additional symmetries would be desirable. Recently we successfully reduced the number of free parameters to 3 in the $K = 2$ model of [10] and, under the 'stronger' symmetry assumptions, to 2 in the $K = 3$ model of [11].

Inspired by the latter two examples we shall now contemplate $K > 3$ channels and try to impose certain symmetry constraints in the manner which could keep our Schrödinger equation (2) with more coupled channels exactly and compactly solvable.

2. PT-symmetry revisited

2.1. Parity re-interpreted as a pseudometric

In the majority of its updated formulations [12–14], $\mathcal{PT}$-symmetric quantum mechanics (PTSQM) replaces the involutive parity $\mathcal{P} = \mathcal{P}^{-1}$ by an invertible and indefinite pseudometric **P**. In the language coined by Ali Mostafazadeh [13] one replaces the $\mathcal{PT}$-symmetry property of the Hamiltonian $H \neq H^\dagger$ by the requirement

$$H^\dagger = \mathbf{P}H\mathbf{P}^{-1}, \qquad \mathbf{P} = \mathbf{P}^\dagger \neq I. \tag{3}$$

It may be understood as a certain necessary condition that the spectra of the observable H and of its 'redundant' conjugate $H^\dagger$ coincide.

A 'hidden' purpose of the postulate (3) lies in the fact that as long as $H \neq H^\dagger$, the standard knowledge of the (presumably, real and discrete) energies E_n and of the related eigenstates $|n\rangle$ of H must be complemented by the *independent* construction of the eigenstates of the conjugate operator $H^\dagger$, i.e., in our adapted Dirac's notation, of the 'ketkets' $|n\rangle\rangle$. Fortunately, in such a situation equation (3) enables us to employ, in the non-degenerate case, the implication

$$H^\dagger|n\rangle\rangle = E_n|n\rangle\rangle \quad \Longrightarrow \quad |n\rangle\rangle = \mathrm{const}(n)\mathbf{P}|n\rangle. \tag{4}$$

The latter condition is of paramount importance for the technical feasibility of the practical applications of the formalism. Indeed, the construction of $|n\rangle\rangle$ becomes straightforward whenever the action of the pseudometric **P** is not too complicated.

A more detailed support of the latter argument may be found, e.g., in the appendices of [10] and in [15] where we studied an application of PTSQM to the Peano–Baker-like two-channel version of the Klein–Gordon equation. On the background of this technical summary we only have to emphasize that the knowledge of the two sets of the vectors $|n\rangle$ and $|n\rangle\rangle$ (forming a biorthogonal basis in our Hilbert space) opens, for all the square-well-type models, the way towards the construction of the necessary 'physical' positive definite metric $\Theta = \Theta^\dagger$. A nice explicit illustration of the recipe (which ascribes an appropriate probabilistic interpretation to the system, cf appendix) has been discussed by Mostafazadeh and Batal [4], with an elegant Krein-space mathematical re-interpretation added recently by Langer and Tretter [8].

2.2. Unitary alternatives to the parity

We feel inspired by the observation that in a perceivable contrast to the physical metric Θ itself, the pseudometric plays just an auxiliary role, via equation (4). In such a context, the requirement of the Hermiticity of **P** is redundant and the main emphasis must be put on its simplicity. This is the key idea of our present paper. In the place of the standard Hermitian pseudometrics **P** we shall try to work with some non-Hermitian pseudoparity operators $\mathbf{R} \neq \mathbf{R}^\dagger$ replacing equation (3) by its alternative

$$H^\dagger = \mathbf{R} H \mathbf{R}^{-1}, \qquad \mathbf{R} \neq \mathbf{R}^\dagger. \tag{5}$$

As long as the Hermitian conjugation is an involution, we may insert equation (5) in its conjugate version $H = [\mathbf{R}^{-1}]^\dagger H^\dagger \mathbf{R}^\dagger$ and arrive at the symmetry requirement

$$H\mathcal{S} = \mathcal{S}H, \qquad \mathcal{S} = [\mathbf{R}^{-1}]^\dagger \mathbf{R}. \tag{6}$$

We believe that its implementation might serve our present purposes.

For the sake of definiteness we shall pay attention to the families of operators **R**,

$$\mathbf{R}_{(K,1)} = \begin{pmatrix} 0 & \dots & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & \dots & 0 & 0 \\ 0 & \mathcal{P} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathcal{P} & 0 \end{pmatrix}, \qquad \mathbf{R}_{(K,2)} = \begin{pmatrix} 0 & \dots & 0 & \mathcal{P} & 0 \\ 0 & 0 & \dots & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \mathcal{P} & 0 & 0 \end{pmatrix}, \dots$$

with the property $\mathbf{R}^{-1} = \mathbf{R}^\dagger$ giving the symmetry $\mathcal{S} = \mathbf{R}^2$. In the representation where we separate the parity, $\mathbf{R}_{(K,L)} = \mathcal{P}\mathbf{r}_{(K,L)}$, we may employ the recurrences for matrices

$$\mathbf{r}_{(K,L+1)} = \mathbf{r}_{(K,1)}\mathbf{r}_{(K,L)}, \qquad L = 1, 2, \dots.$$

At any K the pseudoparities obey the rule $[\mathbf{r}_{(K,L)}]^K = I$ and may be interpreted as finite rotations. At the even $K = 2M$ we get the non-Hermitian $\mathbf{r}_{(K,K-L)} = [\mathbf{r}_{(K,L)}]^\dagger$ at $L = 1, 2, \dots, M-1$. An anomaly occurs at $L = M$ where we note that $\mathbf{r}_{(K,K-L)} = [\mathbf{r}_{(K,L)}]^\dagger = \mathbf{r}_{(2M,M)}$ remains Hermitian. There is no similar Hermitian exception at the odd integers $K = 2M - 1$ with $M > 1$.

We are now prepared to apply the rule (5) to our coupled-channel models (2).

3. The method

3.1. The symmetry-compatible sets of coupling constants

In units $\hbar = 2m = 1$ and in a partitioned matrix notation equation (2) may be reformulated as a diagonalization of the Hamiltonians $H = H_{(\text{kinetic})} + H_{(\text{interaction})}(x)$ where

$$H_{(\text{kinetic})} = \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix}.$$

The separate parity-antisymmetric imaginary square-well potentials (1) will form the array

$$H_{(\mathrm{interaction})}(x) = \begin{pmatrix} V_{Z_{(1,1)}}(x) & V_{Z_{(1,2)}}(x) & \dots & V_{Z_{(1,K)}}(x) \\ V_{Z_{(2,1)}}(x) & V_{Z_{(2,2)}}(x) & \dots & V_{Z_{(2,K)}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ V_{Z_{(K,1)}}(x) & V_{Z_{(K,2)}}(x) & \dots & V_{Z_{(K,K)}}(x) \end{pmatrix} \tag{7}$$

characterized by K^2 different real coupling constants,

$$\mathbf{A} = \begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & \dots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} & \dots & Z_{(2,K)} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \dots & Z_{(K,K)} \end{pmatrix}. \tag{8}$$

As long as the diagonal $H_{(\mathrm{kinetic})}$ commutes with all our pseudoparities, we shall only have to study the consequences of the symmetry (6) upon the variability of the set (8).

In a more detailed analysis of the latter point we must recollect, firstly, the action of parity $\mathcal{P}$ on our Hamiltonian,

$$\mathcal{P}H_{(\mathrm{interaction})}\mathcal{P} = -H_{(\mathrm{interaction})}.$$

This rule enables us to rewrite equation (5) as a condition imposed upon the real matrix of indices (8),

$$\mathbf{A} = \mathbf{r}_{(K,K-L)} \cdot \mathbf{A}^T \cdot \mathbf{r}_{(K,L)}, \tag{9}$$

where ^{T} denotes transposition. This equation is a core of our forthcoming constructions. At any fixed K and L, it must be satisfied as a guarantee that our square-well Hamiltonian H obeys the **R**-pseudo-Hermiticity rule (5). Step-by-step we shall list the solutions of equation (9) distinguishing between the odd K (in a series starting from section 4) and the even K (starting from section 6).

3.2. The determination of the bound-state energies

Our next step may be guided by the elementary single-channel example of [2] with the 'effective' Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) - \mathrm{i}Z_{\mathrm{eff}}\,\mathrm{sign}(x)\varphi(x) = E\varphi(x), \qquad \varphi(-1) = \varphi(1) = 0. \tag{10}$$

At $K = 1$ and $Z_{\mathrm{eff}} = Z$ it has been shown solvable and physical (i.e., possessing the real spectrum) at $|Z_{\mathrm{eff}}| < Z_{\mathrm{crit}} \approx 4.48$ in [2], with a better estimate of $Z_{\mathrm{crit}} \approx 4.475\,308\,56$ derived in [11].

For the generic $K > 1$ and for the negative or positive coordinate x, our coupled set (2) are differential equations with constant coefficients. In a way resembling equation (10) these equations remain solvable by the trigonometric ansatz

$$\varphi^{(m)}(x) = \begin{cases} C_L^{(m)} \sin\kappa_L(x+1), & x \in (-1,0), \\ C_R^{(m)} \sin\kappa_R(-x+1), & x \in (0,1), \qquad m = 1, 2, \dots, K \end{cases} \tag{11}$$

compatible with the 'external' boundary conditions at $x = \pm 1$. We must also impose the $2K$-plet of the standard 'internal' matching conditions in the origin,

$$\begin{aligned} C_L^{(m)} \sin\kappa_L &= C_R^{(m)} \sin\kappa_R, & m &= 1, 2, \dots, K, \\ \kappa_L C_L^{(n)} \cos\kappa_L &= -\kappa_R C_R^{(n)} \cos\kappa_R, & n &= 1, 2, \dots, K. \end{aligned} \tag{12}$$

Their first half may be read as determining, say, the K 'dependent' constants $C_R^{(m)}$ as functions of the K 'independent' constants $C_L^{(m)}$ and of the not yet specified parameters $\kappa_{L,R}$. The ratio of the equations with $m = n$ eliminates all the constants and leads to the single complex condition

$$\kappa_L \operatorname{cotan} \kappa_L = -\kappa_R \operatorname{cotan} \kappa_R. \tag{13}$$

Finally, the insertion of our ansatz (11) in the Schrödinger equation (2) gives

$$\begin{pmatrix} \kappa_L^2 + \mathrm{i}Z_{(1,1)} & \mathrm{i}Z_{(1,2)} & \dots & \mathrm{i}Z_{(1,K)} \\ \mathrm{i}Z_{(2,1)} & \kappa_L^2 + \mathrm{i}Z_{(2,2)} & \dots & \mathrm{i}Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathrm{i}Z_{(K,1)} & \mathrm{i}Z_{(K,2)} & \dots & \kappa_L^2 + \mathrm{i}Z_{(K,K)} \end{pmatrix} \begin{pmatrix} C_L^{(1)} \\ C_L^{(2)} \\ \vdots \\ C_L^{(K)} \end{pmatrix} = E \begin{pmatrix} C_L^{(1)} \\ C_L^{(2)} \\ \vdots \\ C_L^{(K)} \end{pmatrix} \tag{14}$$

at $x \in (-1, 0)$ and the similar K-dimensional diagonalization

$$\begin{pmatrix} \kappa_R^2 - \mathrm{i}Z_{(1,1)} & -\mathrm{i}Z_{(1,2)} & \dots & -\mathrm{i}Z_{(1,K)} \\ -\mathrm{i}Z_{(2,1)} & \kappa_R^2 - \mathrm{i}Z_{(2,2)} & \dots & -\mathrm{i}Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ -\mathrm{i}Z_{(K,1)} & -\mathrm{i}Z_{(K,2)} & \dots & \kappa_R^2 - \mathrm{i}Z_{(K,K)} \end{pmatrix} \begin{pmatrix} C_R^{(1)} \\ C_R^{(2)} \\ \vdots \\ C_R^{(K)} \end{pmatrix} = E \begin{pmatrix} C_R^{(1)} \\ C_R^{(2)} \\ \vdots \\ C_R^{(K)} \end{pmatrix} \tag{15}$$

at $x \in (0, 1)$. For the real energies the latter two sets are just complex conjugates of each other so that we may omit one of them and set $\kappa_R = s + \mathrm{i}t = \kappa_L^*$ with $s > 0$ and $t \in (-\infty, \infty)$.

In the next step generalizing the experience gained in [2, 10, 11] we use another ansatz

$$E = s^2 - t^2. \tag{16}$$

Its use leaves the matrices in equations (14) and (15) purely imaginary so that it is easy to write their common secular equation which is real,

$$\det \begin{pmatrix} Z_{(1,1)} - 2st & Z_{(1,2)} & \dots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} - 2st & \dots & Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \dots & Z_{(K,K)} - 2st \end{pmatrix} = 0. \tag{17}$$

It is to be complemented by the complex equation (13) which, in the same notation, degenerates to the semi-trigonometric *and K-independent* algebraic formula

$$2s \sin 2s + 2t \sinh 2t = 0. \tag{18}$$

We may summarize that the pair of the real algebraic equations (17) and (18) may be expected to specify the real parameters $s = s_n$, $t = t_n$ and the energies $E = E_n$, $n = 0, 1, \dots$ at each particular number of channels K.

As long as equation (18) itself is the same for any K, we may treat it simply as a definition of certain 'universal' curve $t = t_{\text{exact}}(s)$. It is worth noting that its shape carries a lot of resemblance to the half-ovals

$$t_{\text{auxiliary}}(s) = \max(0, -s \sin 2s)$$

(see [2] for a more detailed description of the shape of the exact curve).

The former equation (17) defines, in principle, the K-dependent K-plet of the real eigenvalues $2st \equiv Z_{\text{eff}}$. Thus, once we determine all of them exactly, $Z_{\text{eff}} = Z_{\text{eff}}^{(k)}$,

$k = 1, 2, \ldots, K$, we may interpret all the related physical roots $s = s_n$ and $t = t_n$ (which may be degenerate of course) as the coordinates of the intersections of the above-mentioned universal half-oval curve $t = t_{\text{exact}}(s)$ with any one of the K much more elementary hyperbolic curves $t^{(k)}_{\text{hyperbolic}}(s) = Z^{(k)}_{\text{eff}}\big/(2s)$.

4. The first non-trivial model with odd $K = 3$

4.1. Hermitian choices of pseudometrics P

When we pick up $K = 3$ we have a nice opportunity to distinguish between the parity $\mathcal{P}$, Hermitian pseudometric **P** and its non-Hermitian generalization **R**. In the simplest Hermitian arrangement we may choose the diagonal partitioned operator

$$\mathbf{P} = \mathbf{P}_{(3,0)} = \begin{pmatrix} \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \end{pmatrix} = \mathbf{P}^{\dagger} = \mathbf{P}^{-1}$$

and note that its use in equation (5) does not impose any constraint upon the nine one-parametric square-well interactions

$$V_{Z_{(j,k)}}(x) = \begin{cases} \mathrm{i}Z_{(j,k)}, & x \in (-1, 0), \\ -\mathrm{i}Z_{(j,k)}, & x \in (0, 1), \end{cases} \qquad j, k = 1, 2, 3. \tag{19}$$

All the nine coupling constants $Z = Z_{(i,j)}$ remain independent. This leaves the corresponding solutions very complicated. There are no symmetries in the problem; it must be solved more or less purely numerically. This is, from our present constructive point of view, a not too interesting situation.

The situation is merely marginally improved by the transition to several other, less trivial Hermitian pseudometrics like

$$\mathbf{P} = \mathbf{P}_{(3,1)} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ 0 & \mathcal{P} & 0 \\ \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{P}^{\dagger} = \mathbf{P}^{-1}$$

or

$$\mathbf{P} = \mathbf{P}_{(3,2)} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} \end{pmatrix} = \mathbf{P}^{\dagger} = \mathbf{P}^{-1}.$$

It is easy to show that in the Hermitian cases the partitioned form of the second powers $[\mathbf{P}_{(3,1)}]^2 = [\mathbf{P}_{(3,2)}]^2 = I\mathcal{P}^2$ becomes diagonal.

4.2. Non-Hermitian pseudoparities R

The emerging availability of the first two simplest non-Hermitian pseudoparities should be emphasized at $K = 3$,

$$\mathbf{R} = \mathbf{R}_{(3,1)} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,2)}, \qquad \mathbf{R}^{-1} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \end{pmatrix},$$

$$\mathbf{R} = \mathbf{R}_{(3,2)} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,1)}, \qquad \mathbf{R}^{-1} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = \mathbf{R}_{(3,1)}.$$

In contrast to the Hermitian cases, we only obtain the higher power parallel rules $[\mathbf{R}_{(3,1)}]^3 = I\mathcal{P}^3$ and $[\mathbf{R}_{(3,2)}]^3 = I\mathcal{P}^3$ since $[\mathbf{R}_{(3,1)}]^2 = \mathcal{P}\mathbf{R}_{(3,2)}$ and $[\mathbf{R}_{(3,2)}]^2 = \mathcal{P}\mathbf{R}_{(3,1)}$ in both our genuine non-Hermitian samples.

4.3. The allowed $Z_{(j,k)}$ for the three coupled channels

At the unique index $L = 1$ our **R**-pseudo-Hermiticity condition (9) acquires the linear algebraic form of a set of equations for the couplings $Z_{(j,k)} \equiv a(j,k)$,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \\ a(3,1) & a(3,2) & a(3,3) \end{pmatrix} = \begin{pmatrix} a(1,1) & a(2,1) & a(3,1) \\ a(1,2) & a(2,2) & a(3,2) \\ a(1,3) & a(2,3) & a(3,3) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Due to the non-Hermiticity of the pertaining matrix **r**, these relations represent a much more powerful constraint which leaves just the two coupling constants free and independent. The equations preserve their form under the transposition so that the structure of our $K = 2$ square-well model remains independent of L,

$$\mathbf{A}_{(\text{interaction})} = \begin{pmatrix} Z & X & X \\ X & Z & X \\ X & X & Z \end{pmatrix}, \qquad L = 1, 2. \tag{20}$$

Now we may return to the Hermitian choices of $\mathbf{P} = \mathbf{P}_{(3,1)}$ or $\mathbf{P} = \mathbf{P}_{(3,2)}$ of subsection 4.1 which, obviously, introduced much less symmetry. Indeed, the solution of the corresponding nine linear equations generates just the three non-trivial constraints so that as many as six coupling constants remain independently variable. The same tendency survives at the higher K. We may conjecture that the breakdown $\mathbf{P} \to \mathbf{R}$ of the Hermiticity is connected, definitely, with an enhancement of the symmetry and of the simplicity of the models.

4.4. Energy levels

In the final step of the construction of the bound states at $K = 3$ we may follow either the general recipe of section 3.2 or the recent detailed presentation of the $K = 3$ solutions in [11]. In essence, we have to connect the parameters s and t with the 'effective charge',

$$2st = Z_{\text{eff}}, \qquad Z_{\text{eff}} = Z + F,$$

where, in the notation of equation (20), the three eligible values of the shift $F = F_j$ are to be sought as eigenvalues $[F_1, F_2, F_3] = [2X, -X, -X]$ of the modified matrix (20)

$$\begin{pmatrix} 0 & X & X \\ X & 0 & X \\ X & X & 0 \end{pmatrix}.$$

The three respective eigenvectors may be found in [11]—here our MAPLE software produced their following simpler alternative sample:

$$\{1, 1, 1\}, \qquad \{-1, 0, 1\}, \qquad \{-1, 1, 0\},$$

which is still to be reorthogonalized.

In a way compatible with the results of [11] we may summarize that our $K = 3$ coupled bound states are determined by formulae (11) and (16). The parameters s and t are fixed as intersections of the half-ovals (18) with one of the two available hyperbolic curves,

$$t = t^{(\sigma)}(s) = \frac{1}{2s} Z_{\mathrm{eff}}(\sigma), \qquad \sigma = 1, 2, \qquad Z_{\mathrm{eff}}(1) = Z + 2X, \qquad Z_{\mathrm{eff}}(2) = Z - X. \tag{21}$$

By construction, the second family of intersections represents the twice-degenerate levels.

5. The next model with odd $K = 5$

There is no anomaly in the non-Hermiticity of $\mathbf{r} = \mathbf{r}_{(5,L)}$ with $L = 1, 2, 3$ and 4,

$$\mathbf{r}_{(5,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ldots, \qquad \mathbf{r}_{(5,4)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

All the four different pseudo-Hermiticity conditions (9) with $L = 1, 2, 3, 4$ lead to the same three-parametric coupling-constant matrix

$$\mathbf{A}_{(\mathrm{interaction})} = \begin{pmatrix} Z & X & D & D & X \\ X & Z & X & D & D \\ D & X & Z & X & D \\ D & D & X & Z & X \\ X & D & D & X & Z \end{pmatrix}. \tag{22}$$

Besides its exceptional eigenvector $\{1, 1, 1, 1, 1\}$ pertaining to the obvious eigenvalue $F_0 = 2D + 2X$, the reduced $Z = 0$ form of this matrix is most easily shown to possess the pair of the twice degenerate eigenvalues,

$$F_{\pm} = \tfrac{1}{2}[-D - X \pm \sqrt{5}(-D + X)]$$

with the two respective eigenvectors

$$\left\{ \frac{1}{2} \frac{D - X \pm \sqrt{5}\,(-D + X)}{-X + D}, -\frac{1}{2} \frac{D - X \pm \sqrt{5}\,(-D + X)}{-X + D}, 1, 0, -1 \right\}$$

and

$$\left\{ \frac{1}{2} \frac{D - X \pm \sqrt{5}\,(-D + X)}{-X + D}, -1, 0, 1, -\frac{1}{2} \frac{D - X \pm \sqrt{5}(-D + X)}{-X + D} \right\}.$$

The next step towards the next odd $K = 7, 9, \ldots$ will be discussed in our concluding remarks. Now, for a more specific illustration of some technical subtleties let us return to the systems with the small even numbers of coupled channels.

6. The simplest model with even $K = 2$

It would be easy to relax the involution assumption $\mathcal{P}^2 = I$ as formally redundant and pedagogically partially misleading. Even without such a constraint we get just the most elementary Hermitian option at $K = 2$,

$$\mathbf{P} = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & -\mathcal{P} \end{pmatrix} = \mathbf{P}^\dagger, \qquad \mathbf{P}^{-1} = \begin{pmatrix} \mathcal{P}^{-1} & 0 \\ 0 & -\mathcal{P}^{-1} \end{pmatrix},$$

plus its fully off-diagonal alternative

$$\mathbf{P} = \begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P} & 0 \end{pmatrix} = \mathbf{P}^\dagger, \qquad \mathbf{P}^{-1} = \begin{pmatrix} 0 & \mathcal{P}^{-1} \\ \mathcal{P}^{-1} & 0 \end{pmatrix}.$$

Only for the latter sample choice of the matrix $\mathbf{r} = \mathbf{r}_{(2,1)}$, the compactified version (9) of the pseudo-Hermiticity condition (5) acquires a non-trivial 2×2 matrix form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{pmatrix} = \begin{pmatrix} a(1,1) & a(2,1)) \\ a(1,2) & a(2,2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With involutive $\mathcal{P}$ this model as well as the resulting set of the four equations has already been studied in [10] and may be easily shown to degenerate to the single constraint $a(1,1) = a(2,2)$. Thus, the $K = 2$ version of our present square-well model possesses three free real parameters X, Y and Z:

$$\mathbf{A}_{(\text{interaction})} = \begin{pmatrix} Z & Y \\ X & Z \end{pmatrix}, \qquad L = 1. \tag{23}$$

As long as the pseudoparity remains Hermitian, the model does not fit in the scope of our present paper. Still it exemplifies the general pattern of the construction since both the eigenvalues $F = F_\pm$ of the modified $Z = 0$ version of matrix (23) are easily found, $F_\pm = \pm\sqrt{XY}$ and also the determination of the two respective eigenvectors by our MAPLE program,

$$\{1, \sqrt{X/Y}\}, \qquad \{1, -\sqrt{X/Y}\}$$

is easily verified by hand and tests the recipe.

7. Four coupled channels

Just a smaller representative sample is to be added at $K = 4$, namely, the non-Hermitian

$$\mathbf{R} = \mathbf{R}_{(4,1)} = \begin{pmatrix} 0 & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \end{pmatrix} \neq \mathbf{R}^\dagger, \qquad \mathbf{R}^{-1} = \begin{pmatrix} 0 & \mathcal{P}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{P}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{P}^{-1} \\ \mathcal{P}^{-1} & 0 & 0 & 0 \end{pmatrix}$$

and the 'exceptional' Hermitian

$$\mathbf{R} = \mathbf{R}_{(4,2)} = \begin{pmatrix} 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{R}^\dagger, \qquad \mathbf{R}^{-1} = \begin{pmatrix} 0 & 0 & \mathcal{P}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{P}^{-1} \\ \mathcal{P}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{P}^{-1} & 0 & 0 \end{pmatrix}.$$

It should be noted that while we obtain a diagonal $[\mathbf{R}_{(4,2)}]^2 = I\mathcal{P}^2$ in the second power of our Hermitian operator, and analogous non-Hermitian formula requires the use of the fourth power, $[\mathbf{R}_{(4,1)}]^4 = I\mathcal{P}^4$.

The overall structure of the matrix of couplings **A** compatible with our requirement of the 'maximal' non-Hermitian symmetries (9) ceases to be unique at $K = 4$. *A priori*, this follows from the observation that besides the matrix-transposition mapping between the pseudo-Hermiticity constraints at $L = 1$ and $L = 3$, one also encounters the anomalous Hermitian problem at $L = 2$. In the latter case we have to solve the 16 linear algebraic equations

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a(1,1) & a(1,2) & a(1,3) & a(1,4) \\ a(2,1) & a(2,2) & a(2,3) & a(2,4) \\ a(3,1) & a(3,2) & a(3,3) & a(3,4) \\ a(4,1) & a(4,2) & a(4,3) & a(4,4) \end{pmatrix}$$
$$= \begin{pmatrix} a(1,1) & a(2,1) & a(3,1) & a(4,1) \\ a(1,2) & a(2,2) & a(3,2) & a(4,2) \\ a(1,3) & a(2,3) & a(3,3) & a(4,3) \\ a(1,4) & a(2,4) & a(3,4) & a(4,4) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which only impose the six constraints upon the 16 couplings. From the practical point of view, too many of them remain freely variable. Similar results are also obtained for the other Hermitian operators **R**.

In contrast, the parallel and transposition-related $L = 1$ and $L = 3$ non-Hermitian versions of equation (5) give the set of 16 equations

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{A} = \mathbf{A}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = 1$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{A} = \mathbf{A}^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad L = 3,$$

respectively. Both of them give *the same* four-parametric set of coupling constants

$$\mathbf{A}_{(\text{interaction})} = \begin{pmatrix} Z & U & D & U \\ L & Z & L & D \\ D & U & Z & U \\ L & D & L & Z \end{pmatrix}, \qquad L = 1, 3, \tag{24}$$

compatible with both our non-Hermitian pseudoparities and the four quadruplets of the independent elements sitting on the main diagonal (Z), side diagonals (D) and in an upper square (U) and lower square (L). After an obvious permutation of the basis this matrix may be understood as a partitioned structure

$$\mathbf{A}^{(\text{permuted})}_{(\text{interaction})} = \left(\begin{array}{cc|cc} Z & D & U & U \\ D & Z & U & U \\ \hline L & L & Z & D \\ L & L & D & Z \end{array}\right), \qquad L = 1, 3, \tag{25}$$

which represents a partitioned generalization of the 2×2 model (23). Directly, this solution may be derived from the pseudoparity

$$\mathbf{r}^{(\text{permuted})} = \mathcal{P}^{-1}\mathbf{R}^{(\text{permuted})} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which is just a permuted version of the above non-Hermitian square root $\mathbf{r}_{(4,1)}$ of the unit matrix.

Four eigenvalues $F_j = [-D, -D, D+2\sqrt{UL}, D-2\sqrt{UL}]$ of the modified matrix (24) with $Z = 0$ form the partially degenerate quadruplet. The respective eigenvectors read

$$\{1, 0, -1, 0\}, \quad \{0, 1, 0, -1\}, \quad \left\{1, \frac{\sqrt{UL}}{U}, 1, \frac{\sqrt{UL}}{U}\right\}, \quad \left\{1, -\frac{\sqrt{UL}}{U}, 1, -\frac{\sqrt{UL}}{U}]\right\}.$$

The quadruplet of eigenvalues derived from the alternative 4×4 matrix (25) with $Z = 0$ remains unchanged of course. Even the respective eigenvectors themselves become merely predictably influenced by the underlying permutation,

$$\{0, 0, 1, -1\}, \quad \{-1, 1, 0, 0\}, \quad \left\{1, 1, \frac{\sqrt{UL}}{U}, \frac{\sqrt{UL}}{U}\right\}, \quad \left\{1, 1, -\frac{\sqrt{UL}}{U}, -\frac{\sqrt{UL}}{U}\right\}.$$

8. Six coupled channels

While the choice of the Hermitian pseudoparity $\mathbf{r}_{(6,3)}$ is not sufficiently restrictive and leaves 21 free parameters in the related 6×6 coupling-matrix $\mathbf{A}$, much more symmetry (with just seven free parameters) is induced by all the non-Hermitian $\mathbf{r}_{(6,L)}$ with $L \neq 3$.

It is worth noting that different patterns are obtained at $L = 1$ or $L = 5$ (when the resulting real matrix $\mathbf{A}$ remains asymmetric) and at $L = 2$ or $L = 4$ (when the resulting real matrix $\mathbf{A}$ becomes symmetric). In the former case, a suitable permutation of the matrix indices leads to a maximally compact picture,

$$\mathbf{r}^{(\text{permuted})}_{(6,1)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{A}^{(\text{permuted})}_{(\text{interaction})} = \left(\begin{array}{cc|cc|cc} Z & Y & G & B & F & B \\ X & Z & C & F & C & G \\ \hline F & B & Z & Y & G & B \\ C & G & X & Z & C & F \\ \hline G & B & F & B & Z & Y \\ C & F & C & G & X & Z \end{array}\right). \tag{26}$$

The partitioning of the latter asymmetric matrix indicates how our system may be visualized as a coupled set of its three two-dimensional asymmetric subsystems of the form (23).

For the second option with $L = 2$ it is remarkable to note that while the former pseudoparity remains asymmetric and, hence, non-Hermitian, the latter matrix of coupling constants $\mathbf{A}$ appears to be, for some unknown reason, symmetric:

$$\mathbf{r}_{(6,2)}^{(\text{permuted})} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{A}_{(\text{interaction})}^{(\text{permuted})} = \left(\begin{array}{ccc|ccc} Z & X & X & C & D & G \\ X & Z & X & G & C & D \\ X & X & Z & D & G & C \\ \hline C & G & D & A & B & B \\ D & C & G & B & A & B \\ G & D & C & B & B & A \end{array}\right). \tag{27}$$

Moreover, its inspection reveals that it is again tractable as a coupled system of its two three-dimensional subsystems of the symmetric-matrix form (20).

One has to add that the diagonalization of the $Z = 0$ version of the matrix (26) is still feasible and gives the two non-degenerate eigenvalues

$$F_{\pm 0} = G + R \pm \sqrt{2CY + 2BX + XY + 4BC}$$

and the two doubly degenerate eigenvalues

$$F_{\pm 1} = -\tfrac{1}{2}G - \tfrac{1}{2}R \pm \tfrac{1}{2}\sqrt{-3G^2 + 6GR - 3R^2 - 4BX + 4XY + 4BC - 4CY}.$$

In contrast, the study of model (27) is hindered by the occurrence of the two different couplings on the main diagonal. One must employ a shift of the eigenvalues $F = \lambda + (Z + A)/2$ which leads to the diagonalization of the matrix of the form

$$\begin{pmatrix} -\omega & X & X & C & D & G \\ X & -\omega & X & G & C & D \\ X & X & -\omega & D & G & C \\ C & G & D & \omega & Y & Y \\ D & C & G & Y & \omega & Y \\ G & D & C & Y & Y & \omega \end{pmatrix}$$

with non-vanishing main diagonal, $\omega = (Z - A)/2$. Still, in a way resembling the previous $L = 1$ model we get the two non-degenerate eigenvalues

$$\lambda_{\pm 0} = Y + X \pm \sqrt{\Lambda_0}$$

with the abbreviation

$$\Lambda_0 = Y^2 - 2XY + X^2 + \omega^2 + 2Y\omega - 2X\omega + D^2 + 2DC + C^2 + 2GD + G^2 + 2GC,$$

accompanied by the two doubly degenerate eigenvalues

$$\lambda_{\pm 1} = -\tfrac{1}{2}Y - \tfrac{1}{2}X \pm \tfrac{1}{2}\sqrt{\Lambda_1},$$

where the discriminant Λ_1 is equal to the sum

$$Y^2 - 2XY + X^2 + 4D^2 + 4\omega^2 - 4Y\omega + 4X\omega - 4DC + 4C^2 - 4GD - 4GC + 4G^2.$$

9. Concluding remarks

9.1. Seven and more coupled channels at odd $K = 2M - 1$

The pattern initiated by $K = 3$ and $K = 5$ is perpetuated at the next odd dimension $K = 7$ where the linear system of 49 equations yields the four free parameters in the L-independent solution

$$\mathbf{A}_{(\text{interaction})} = \begin{pmatrix} Z & X & Y & D & D & Y & X \\ X & Z & X & Y & D & D & Y \\ Y & X & Z & X & Y & D & D \\ D & Y & X & Z & X & Y & D \\ D & D & Y & X & Z & X & Y \\ Y & D & D & Y & X & Z & X \\ X & Y & D & D & Y & X & Z \end{pmatrix}. \qquad (28)$$

It is easy to see the general pattern and to guess the structure of $\mathbf{A}_{(\text{interaction})}$ at any higher odd dimension $= 2M - 1$. At any M this hypothesis may be verified, say, in MAPLE, by using the same algorithm as employed in the above calculations performed at the first three non-trivial indices $M = 2$, $M = 3$ and $M = 4$.

In a way extrapolating the $K = 5$ results we may move to $K = 7$ in equation (28) and guess the exceptional eigenvector $\{1, 1, 1, 1, 1, 1, 1\}$ and its eigenvalue $F_0 = 2D + 2X + 2Y$. The three further eigenvalues of the matrix

$$\begin{pmatrix} 0 & X & Y & D & D & Y & X \\ X & 0 & X & Y & D & D & Y \\ Y & X & 0 & X & Y & D & D \\ D & Y & X & 0 & X & Y & D \\ D & D & Y & X & 0 & X & Y \\ Y & D & D & Y & X & 0 & X \\ X & Y & D & D & Y & X & 0 \end{pmatrix}$$

are doubly degenerate and may be written in the form of Cardano formulae. They represent the real roots in terms of the complex quantities and, for this reason, we omit them here and leave their generation to the interested readers as an easy exercise.

9.2. Eight and more coupled channels at even $K = 2M$

One gets lost when solving equation (9) in the Hermitian case at $L = 4$ with 37 free parameters in $\mathbf{A}_{8,4}$, 29 of which occur there in pairs.

Similarly, not enough symmetry is induced by the $L = 2$ pseudoparity since the resulting asymmetric $\mathbf{A}$ depends on as many as 16 free parameters, each occurring strictly four times.

Thus, the only satisfactory reduction is obtained from the non-Hermitian pseudoparity $\mathbf{r}_{(8,L)}$ with $L = 1$ (and, identically, $L = 3$ etc). The explicit solution of the corresponding equation (9) leads to the permission of the eight free coupling strengths, each occurring eight times in $\mathbf{A}$. One parameter sits simply on the main diagonal and one on the two 'submain' diagonals of the two off-diagonal quadrants. The remaining six octets form the asymmetric pattern of three doublets reflected pairwise by the main diagonal. The first pair jumps over the main diagonal and the second one over the two submain diagonals while the last pair simply fills the remaining vacancies. We may again separate the even and odd rows and columns and obtain a rearranged matrix $\mathbf{A}$ which is partitioned in the four submatrices where the two diagonal ones are composed of the four parameters only and represent just a transposition of each other. Each of the two off-diagonal blocks depends just on the two parameters in a way resembling slightly the partitioned structure of equation (25).

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Appendix. A note on the interpretation of the Hamiltonians

In a less usual approach to quantum mechanics as outlined in review [3] the Hamiltonian is allowed to be quasi-Hermitian, i.e., Hermitian with respect to some 'anomalous', non-trivial metric $\Theta \neq I$ in the Hilbert space of states. Of course, one must, first of all, redefine the new, 'anomalous' scalar product

$$|a\rangle \odot |b\rangle \equiv \langle a|\Theta|b\rangle, \qquad \Theta = \Theta^\dagger > 0. \tag{A.1}$$

In effect, this is equivalent to a replacement of the standard Hilbert space $\mathcal{H}$ by its new and Θ-dependent 'physical' version $\mathcal{V}$ equipped with the more flexible and adaptable product (A.1).

The key point is that once we stay within the innovated space $\mathcal{V}$, all the basic principles of quantum mechanics remain unchanged. At the same time, the new flexibility carried by our freedom of the choice of Θ is compensated by the loss of the meaning of the standard Hermitian conjugation $H \to H^\dagger$. Indeed, the standard textbook Hermiticity of the observables $A = A^\dagger$ must be replaced by the requirement

$$|Aa\rangle \odot |b\rangle \equiv |a\rangle \odot |Ab\rangle,$$

which may be reread as an obligatory quasi-Hermiticity property

$$A^\dagger = \Theta\, A\, \Theta^{-1} \tag{A.2}$$

of all the observables A in the new formalism. Of course, no new physics is being discovered in this manner because all the operators A with property (A.2) are simply Hermitian with respect to the *fixed* new metric $\Theta \neq I$ [3] denoted usually as η_+ by Mostafazadeh [13] and factorized as $\Theta = \mathbf{CP}$ with 'charge' $\mathbf{C}$ by Bender *et al* [14].

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